Coordinating Units Notes for Video Group 1
E. Yackel

Notes For Coordinating Units Video Group 1

The Coordinating Units (CU) instructional sequence is designed to foster children's development of the initial conceptual foundations for the base-10 place value system. It provides the basis for reasoning that involves coordinating units of ten with units of one. It is important for teachers to understand that developing the *conceptual foundation* for coordinating units of tens and ones is a separate matter from, though related to, using the conventional notation of the standard US algorithms for addition and subtraction. (As an aside, we note that in some other countries other algorithms are used as well.) Traditional US instruction has not always made this distinction with the result that teachers sometimes view children's learning to coordinate units as synonymous with developing proficiency with the standard US algorithms. In this video group and these accompanying notes we clarify this distinction. In fact, we devote the first portion of this video group and these notes entirely to instructional activities designed to foster the basic understandings of coordinating units of tens and ones. Only after that, will we address instruction for adding and subtracting two-digit numbers based on coordinating units of tens and ones.

Sequence Overview

This sequence begins with students engaging in activities that involve a *single* quantity. Students are presented with a quantity either all in ones or in tens and ones and they transform that quantity by either packing or unpacking. The activities shift from students packing and unpacking using the physical materials of Unifix cubes to reasoning from visual imagery. As the sequence progresses, the activities involve *changing the quantity* by either increasing or decreasing the initial amount. Here again, students begin working on these activities by using Unifix cubes to enact the changes and then progress to reasoning from visual imagery.

Teachers that have used the Structuring Numbers instructional sequence will notice similarities in how the two sequences progress. In both sequences students work with a single quantity in Part I of the sequence and then in Part II shift to working with tasks that involve changing the initial quantity by adding more or taking some away. Likewise, within each of Parts I and II, students shift from using physical materials, the arithmetic rack in the case of the Structuring Numbers sequence and Unifix cubes in the case of the Coordinating Units sequence, to reasoning from visual imagery.

Part I: Becoming Efficient With Reasoning About Coordinating Units With A Single Quantity

Activities in Part I of the Coordinating Units sequence involve reconceptualizing a quantity by shifting between units of ten and units of one, as appropriate. There are five activities in Part I. They are:

- Introducing the candy shop scenario
- Packing candies
- Circling candies

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- Candy flash
- Transformation activity

Some of these activities will be used in the classroom only once or twice while others will be used often. We will describe the activities in detail and explain different things children do. In doing so, we attempt to clarify the conceptual demands of the activities and how they function to help children advance in their understanding of coordinating units.

Introducing the Candy Shop Scenario

The Coordinating Units instructional sequence is based on what is called Realistic Mathematics Education instructional design theory. Accordingly, it starts with a scenario that students can use as the basis for personally meaningful activity.[1]

Scenario

Here I describe one way to introduce the scenario.

A woman named Mrs. Wright makes and sells candies in her home. When she first opens her business, customers stop by and buy a small number of candies. Maybe they buy 5 chocolate candies, or 3 orange candies, or 4 peppermint candies. People like her candies so much that soon they come to buy a lot of candies. They might buy 65 chocolate candies, or 32 orange candies, or 28 lemon candies. Business is so good that Mrs. Wright opens a Candy Shop.

Mrs. Wright's Candy Shop

But she has a problem. The customers must stand in line a long time while Mrs. Wright counts out the candies for the customers. It takes her a long time to count out 65 candies or 32 candies or 28 candies. Her customers must stand in line so long. Mrs. Wright realizes that she has a problem. What can she do to solve her problem? (At this point the teacher invites students to make suggestions for how Mrs. Wright might solve her problem. Typically, suggestions include that she hire help or that she count out and prepackage candies in advance.)

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[1] Viewers interested in more detail about the design theory and the role of the scenario are referred to the brief note, "Understanding the design principles of the coordinating units instructional sequence," that is available on this website.

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The scenario continues by explaining that Mrs. Wright decides to count out and package up candies in advance. She decides to pack up candies into rolls. Now she has to decide how many candies to pack into each roll. The reason for packing candies into rolls is to make it easier/faster (more efficient) for her to get out the number of candies the customers asked for—so that she doesn't have to count the candies one by one. (At this point invite students to make suggestions for how many candies Ms. Wright might pack in each roll. Typically suggestions include 2, 5, 20, 100, 10, and other numbers.)

The scenario continues by explaining that what Mrs. Wright decides to do is to pack 10 candies in each roll. She chooses 10 in each roll *because* she knows how to count by 10's.

There are many ways to introduce the scenario. I've given one example. Each teacher will use his or her own ideas to make the story real to the students. It is important to take sufficient time to make the scenario as realistic as possible.

The purpose of a scenario is to create a realistic situation within which students can reason. Students can imagine working for Mrs. Wright. They can easily imagine how they would go about packing up the candies into rolls in her shop when more are made or unpacking, as needed, to sell to a customer. In this way, the story creates the background for the physical activity the children engage in, which eventually evolves into reasoning about quantities. And as a result, we will not need to give procedures for the children to follow. The scenario is introduced one time, but many teachers review it briefly for a few days to ensure that it is familiar to everyone in the class.

I want to emphasize that by the end of the introduction all students should understand that there are 10 candies in each roll. In introducing the scenario, it is important to solicit several answers from students about how many they think might be in a roll and their reasons for choosing their amounts. The purpose for inviting children's input is not to see if they can "guess" 10 candies in each roll, but is to get them to feel invested in the story line. As we said before, the reason Mrs. Wright puts 10 candies in a roll is because *Ms. Wright knows how to count by 10's*. So that makes it easier for her to count out large quantities.

Physical materials to accompany the scenario

The *physical materials* students use with this instructional sequence are Unifix cubes. The teacher explains that these cubes are "pretend" candies in Mrs. Wright's shop. The feature that makes Unifix cubes ideal is that individual cubes (pretend pieces of candy) can easily be snapped together to create rolls. So students can be workers in Mrs. Wright's candy shop. They can snap 10 pieces together to create a roll. In other words, they can physically change 10 units of one into one unit of 10. Likewise, they can unpack a roll, if needed, by breaking a roll into individual pieces. In other words, they can physically change one unit of 10 into 10 units of one. It is for this reason that we recommend against using materials that are typically called "base ten blocks." With those materials 10 individual ones can be lined up but the result is a line of ten individual things. It is not *one thing*, one entity. You cannot pick it up and hold it like you can a stick of ten, a roll made by using Unifix cubes.

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Classes we have worked in typically have a supply of 50 Unifix cubes per student, kept in Ziploc bags of 50. That way each child can have a bag of 50 cubes to use to solve tasks. It is helpful if all of the cubes in a bag are the same color so that when they are packed up there is a unity to the physical appearance of a roll, once packed up. When tasks involve quantities beyond 50, two students can work together and share their cubes to carry out the task. We recommend that the cubes be in sticks of 10 within the bags. That way, children can easily access rolls without having to pack them up each time. Finally, we mention a pragmatic matter for the teacher's benefit. Every time students put their materials away, they should make sure that the cubes are packed into tens. That way the teacher doesn't have to pay attention to that detail continually.

As we discuss the instructional activities, you will see that the first activity, packing candies, is an exception to what I have said above. For that activity, students start with loose candies that are not packed up and students, working in pairs, have different numbers of cubes in their bags.

Packing Candies

The first activity after introducing the scenario is one we call "**packing candies**." It is designed to focus children's attention on the difference between thinking in terms of ones (individual candies) and thinking in terms of tens (rolls). Children work with a partner. Each pair of students is given a bag of loose Unifix cubes and they are told that these are pretend candies. They are told that they are workers for Mrs. Wright and are to pack up these candies into rolls. But before they begin, they should first estimate (guess) how many rolls there will be. As they begin packing, they might decide to revise their estimates. An important aspect of this activity is that students are not told how many loose candies are in the bag they are given. They are expected to make an estimate of the number of rolls based on just looking at the bag.

I want to clarify here that the intention is that the various bags of loose candies given to students are not all the same. Some might have 58 candies, some 31, and some 98. In other words, some will appear to have only a few candies while others will appear to have many candies. Students are aware of whether they have only a few or have many. After students have finished packing their candies into rolls, there is a class discussion in which each group shares their initial estimate of how many rolls they would get and how many rolls they really did get when they packed them up. For this discussion to be meaningful it is important that as they share with the class, the children display their packed up rolls for others to see. I want to explain what I mean by this and why it is important.

As an example, suppose Joe and Sam had a bag with 46 loose candies. (Of course, these boys didn't know how many there were to begin with.)

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After packing up, their desk looks like this.

When it is their turn to share, they show the class what they have and tell what their estimates were. Next, they have to tell the class how many rolls they have after packing. What would we expect them to say? You and I can see that they have 4 rolls, but in our experience, it is not necessarily obvious to Joe and Sam. In fact, it is quite common in this situation for the boys to report that they have 40 rolls and for many in the class to agree.

What is going on here? Why do they say 40 and not 4? These boys are not making a distinction between the number of rolls and the quantity of candies in those rolls. They are not distinguishing between the *number of tens* and the *quantity in those tens*. An intervention at this point, either by the teacher or another student might be to point to one roll and say, "How many *rolls* is this?" Now Joe and Sam might reply, "ten." These boys are answering the question, "How many is this?" in terms of how many candies, that is, how many *ones*. They are not yet differentiating between how many ones (pieces of candy) and how many tens (rolls). And that is precisely the purpose of this task—to call attention to this difference. Notice how the scenario permits us to talk about pieces of candy (pieces) and rolls rather than ones and tens. That is an advantage because the words *pieces of candy* and *rolls* permit us to avoid using numerical quantities in the conversation. It is less confusing to talk about how many *rolls* than to talk about how many *tens*. Even so, the issue goes beyond the use of vocabulary—of which words we are using. It is a conceptual issue and precisely the issue this activity is intended to bring to the forefront.

We know from extensive experience in many first grade classrooms that the discussion about how many rolls is often a protracted one. Even after agreeing that Sam and Joe have 4 rolls (and not 40 rolls), the same error is likely to come up again and again as other groups display their packed up candies and talk about how many rolls they have. Teachers new to this activity are often amazed at how pervasive this misunderstanding is. We want to emphasize that this is a conceptual issue and one that takes time for children to develop. It is not just a slip of the tongue. In fact, being able to *coordinate tens and ones* is precisely the goal of this sequence so it is important for teachers to move on to the next activities that continue to work to develop these basic understandings. The intention is that this activity is to be used only one time. The class will have many discussions that involve distinguishing between the number of rolls and the total quantity in those rolls in the activities that follow.

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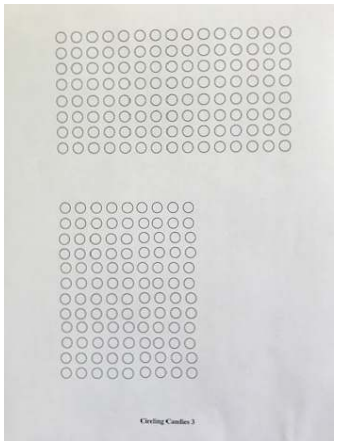
I want to emphasize that the purpose of the packing candies activity is to provide a situation where students are presented with a quantity in units of one and they are asked to reorganize it into units of ten. *The students physically change the materials from units of one to units of ten. In other words, they actually create units of ten from units of one.* They physically make *a roll (a ten)* from *10 loose candies (ones).* As the sequence progresses, students will get to the point of *conceptually* making a ten from 10 ones. At this stage however, it is done *physically*—with materials.

I want to add three notes to teachers about this activity. First, it is essential that this task be about the number of *rolls* and not the number of candies. In our experience teachers sometimes misunderstand this task and mistakenly ask children to estimate the number of candies. But of course, that completely misses the purpose of this task. So teachers need to be careful to avoid this error. Second, it is important that all of the Unifix cubes in the loose bag are of the same color. The reason is simple. When all of the Unifix cubes a pair of students has are the same color, the rolls they get will have the appearance of a *unity*. If a roll would consist of a variety of different colors, the sense of *one thing* is lost. Third, it is important that the cubes used actually snap together. Unifix cubes are ideal for this purpose. Sets of materials typically called base ten blocks are inappropriate because while ten individual ones can be lined up together, they do not actually snap together to create a single entity, a ten.

Circling Candies

The next instructional activity, **circling candies**, is similar in some ways to the prior packing candies activity. Here students see a "picture" of a number of individual candies (shown in an array) and their task is to first estimate the number of rolls there would be when the candies are packed up and then, after recording their estimate, to circle groups of candies to show packing them into rolls. Here you see an example of a page students are given to work on. As you can see, there are two separate tasks on this page.



Example of student page for Circling Candies

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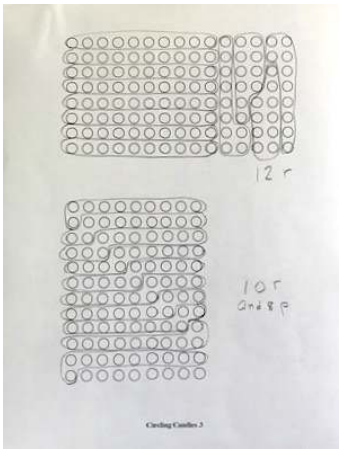
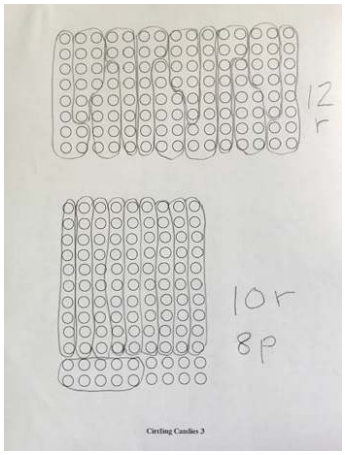
The circling candies and packing candies activities begin the same way. Students are presented with materials in units of one—individual candies. They begin by estimating how many rolls there would be if the candies were packed up. The difference between the two activities lies in what happens next. In the packing candies activity students use Unifix cubes as pretend candies to *physically* create rolls. They physically change ten ones into one ten. Here, students circle the candies to show which candies they *would* pack into a roll.

I want to emphasize again that as with the packing candies activity, it is important that the teacher not misinterpret this and ask students to estimate how many candies there are. The entire purpose of this activity is to get students to reconfigure a set of ones into sets of tens, that is, to reconfigure a bunch of individual pieces of candy into rolls of candy, and to think about the number of rolls there are.

As a pragmatic point we note that it is advisable to have children work alone rather than with a partner on the circling candies tasks. The reason is simple. It is virtually impossible to share a page like this one. Each child will want to do the circling so why not give them the chance to do so.

Here you see two different ways children completed the tasks on this page.





Teachers often choose someone's work to display to the class to initiate the subsequent class discussion. The focus of the discussion is not on how the circling was done but rather on how many rolls there are after packing. And once again, the issue that arises in the discussion is to clearly distinguish the *number of rolls* from the *quantity in the rolls.*

In the second task on this page there are 10 rolls after packing, but it is not uncommon for more than a few children to say that there are 100 rolls. One approach teachers can use is to say, "Let's count how many rolls there are." And then count together as a class, "one roll, two rolls, three rolls, four rolls, five rolls, six rolls, seven rolls, eight rolls, nine rolls, ten rolls" as the teacher points to the individual rolls. "And there are eight pieces left over." Then say, "Now let's count how many candies there are and count together as a class, "ten,

twenty, thirty, forty, fifty, sixty, seventy, eighty, ninety, one hundred" as the teacher points to the individual rolls and then continuing 101,102 103, 104, 105, 106, 107, 108 as she points to the remaining 8 loose pieces.

I want to emphasize something here that many first grade teachers already know, and that is that children can often *say* the number word sequence, "ten, twenty, thirty, forty, etc." without knowing that the difference between 10 and 20 is 10 or that the difference between 20 and 30 is 10. Saying the number word sequence might be a rote activity for them. It might be like reciting a poem but it might not have any quantitative meaning. We know this because if we ask a first grader, "How far is it from say 20 to 30?" the child might have to actually count 21, 22, 23, 24, … 30 while keeping track on his fingers to know that it's 10. These early activities in this coordinating units sequence that involve reorganizing items presented in ones into units of ten are important to help students develop these understandings. Therefore, we have to be careful to not rush through these activities too quickly. They seem so obvious to us as adults. Even so, we encourage teachers to continue to push forward to other activities in the sequence. We recommend spending one or two lessons but not more on the circling candies activity. The materials provided include a total of four "circling candies" pages with two tasks per page for student use.

Candy Flash

The next instructional activity is one we call "**candy flash**." Most teachers love this activity. As they use it repeatedly, they can see how children advance in the way they reason and how it contributes to their understanding of coordinating units. This activity will be used productively for many weeks, including in combination with activities from Part II of the sequence.

In this activity the teacher displays a configuration of rolls and pieces using the flashing approach. Students figure out the quantity of candy shown and tell how they figured it out. Before going further, let me describe the flashing approach for those that are not yet familiar with it. It's an approach we use with many of our activities. The teacher displays something, in this case a configuration of rolls and pieces, for a few seconds, hides the display for several seconds, shows the display again for several seconds, hides again for several seconds, and repeats the process up to three or sometimes four times.

Before we show an example, we have to explain that in the pictures we will show, a roll and a piece look like this.

Roll Piece

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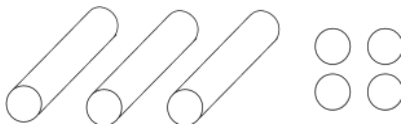
In our experience it is important to use a few introductory examples to ensure that children understand the intention of the task. After establishing what a roll will look like and what a piece of candy will look like we might use this as our first task.



Possible Initial task

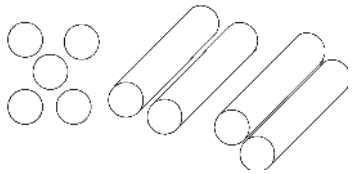
"These are the candies Mrs. Wright has." Flash this image three or four times, as described earlier and ask the students, "How many candies does Mrs. Wright have?" Leave the image off as children give their answers and turn the image on again as the discussion proceeds so children can make sense of what others say.

The purpose of this task is to establish what we mean by the question, "How many candies does Mrs. Wright have?" The answer is 34. In other words, we are not asking students to report that there are 3 rolls and 4 pieces. We are asking about the total quantity. Children have to recognize which items represent rolls (units of ten) and which represent pieces (units of one) and they have to be able to coordinate those two different types of units to figure out the quantity—to figure out that if you put 3 tens together with 4 ones you get 34.

This task is easy for someone that already knows that 3 tens together with 4 ones is 34. But, children that do not yet just know that have to figure it out somehow. How might they do that? One way that we often see is that they count. For example, they might begin counting at the left, "10, 20, 30" and continue on as they move to the pieces on the right. In most classrooms there will be one or two children that come up with 70 candies as an answer. What is happening here? Such children start counting by tens, "10, 20, 30" and continue on by tens, "40, 50, 60, 70" as they count the pieces. These children are unable to make a shift *in the midst of their counting activity* from counting by tens to counting by ones. In some cases, these children's counting activity is entirely rote. It has nothing to do with quantity. It is more like reciting a poem. So after they begin saying, "10, 20, 30" they continue on with the *tens poem*.

Teachers that have not previously used this approach to coordinating units instruction might think that I am making this up. No one would say that there are 70 candies here. But I can guarantee you that I have often seen children do exactly that. Other children will disagree with 70 and will say, "How did you get 70?" And a lively discussion will ensue. In the discussion, children will explain to each other ways they got the answer of 34 and why 70 is not correct. Class discussions like this are important because it is through the discussions that children have to justify their answers to others in the class and, as they hear others' responses to what they say, they have the opportunity to advance in their understandings.

Here is a task we might use next.



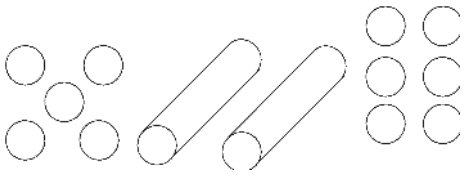
Next Task

Again, the image is flashed 3 or 4 times and children are asked how many candies Mrs. Wright has. And again, the image is left off as children begin to give their answers. It is important for the image to be visible again as the discussion progresses so children can refer to it.

As with the previous task, some children will find this task easy but others will count, and this time some will answer that there are 9 candies—those children began counting by ones as they counted the pieces on the left and continued counting by ones throughout. Someone might say, "You should always start with the tens" but, of course, that isn't necessary at all. What is important is that children *not* develop habits that are based on visual cues rather than on quantitative meaning. And, that is the reason we include this task. In many traditional instructional materials, tens are always presented on the left with the ones on the right. When that is done consistently, some children will base their reasoning on what is on the left side of the picture and what is on the right side of the picture rather than on the quantitative value of various parts of the picture. So we include this example so that teachers understand that it is the quantity represented by the image shown and not its position in the picture that is important for children to attend to.

After these introductory tasks, which will be used only once, we encourage teachers to use tasks where the rolls and pieces are not always placed together, and to always include 10 or more pieces. Here is an example.

Example 1:



The number of candies in this picture is 31. And that is what children are expected to figure out. If a child says, "I saw 2 rolls and 11 pieces," or maybe "I saw 5 pieces, 2 rolls, and 6 more pieces," the teacher should ask, "So how many candies is that? Do you have a way to figure that out?"

What do children do to figure this out?
Lourdes might say, "I saw 2 rolls and that's 20 and then I counted the pieces, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31." Notice several things about her solution. First, she distinguished

between the tens and the ones—the rolls and the pieces—and knew their quantitative value. Second, she didn't need to count the rolls by saying, "10, 20." Rather she said, "I saw 2 rolls and that's 20." Finally, she counted the pieces by ones. Presumably had there been only the 5 pieces on the left or the 6 pieces on the right, but not both, she could have easily replied, "25 candies" or "26 candies." But the complexity of the picture requires her to deal with the 5 pieces and the 6 pieces, which exceed 10, and also with the 2 rolls. Her approach was to count on by ones from 20 to figure out how many in all. As she explains her solution to the class, she most likely will indicate which pieces she counted first and which next. If she doesn't, others will probably ask her.

An important teacher move at this point is to ask, "Did anyone figure it out without counting?" Danielle might say, "I took four pieces from the left side and put them together with the 6 pieces on the right side to make another roll. Now I have 3 rolls, that's 30, and I have 1 piece left over so it's 31." Notice how the language of the scenario has come into play in Danielle's solution. Also notice that Danielle did not say, "I thought about packing up the candies in Mrs. Wright's shop to get another roll." She said I *made* another roll. Well, she wasn't touching any physical materials so what we know is that she made the roll in her mind. She was reconceptualizing 10 units of one (10 pieces) as one unit of ten (a roll). If we think back to the first two activities in this sequence, packing candies and circling candies, we can see the progression. In packing candies children physically make a roll by combining 10 loose pieces of candy. The result is that 10 things became one thing. In the circling candies activity children draw a loop around 10 candies in a picture to show what they *would* pack. And here we see Danielle making a roll entirely mentally. Now I want to emphasize that not all children make this shift to making rolls mentally quickly. Some children will continue to use counting approaches for some time to figure out how many candies there are. But through discussions in which children give solutions like Danielle did, others begin to be able to do so also.

Next, we consider several other solutions students might give that are similar in nature to Danielle's solution. Max might say, "I took 5 pieces from the 6 on the right and put them with the 5 on the left to get another roll. I had 3 rolls and 1 piece left over. That's 31." Samantha might say, "I saw 5 pieces and 6 more pieces. That's 11 pieces so I made another roll and had 1 piece left over. Now I have 3 rolls and 1 piece. That's how I got 31."

In each of the solutions given by Danielle, Max and Samantha the children thought about the pieces shown and reconceptualized them as a roll and one piece (one left over piece). Further, each of them *said* that they now had 3 rolls. In other words, they mentally transformed the picture that showed 2 rolls and 11 pieces into one with 3 rolls and 1 piece. They coordinated the units of one with the units of ten.

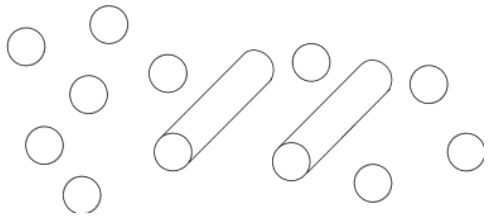
Another way to say it is that they had a configuration of tens and ones and they transformed it into a different configuration of tens and ones that represented the same quantity. Or yet another way to say it is that there is more than one way to use units of ten and units of one to show a given quantity. And that is going to lead us to the next instructional activity, one we call the transformation activity. But before we discuss that

activity we want to make some important pragmatic suggestions about this candy flash activity.

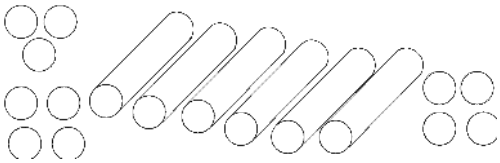
Important considerations for candy flash activity

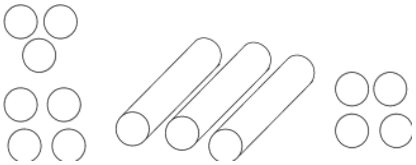
There are a number of important considerations the teacher must attend to when using the candy flash activity.

1. The number of individual pieces should always be 10 or greater. The reason is that when there are fewer than 10 pieces, children can figure out the answer without having to reconceptualize a group of pieces as a roll and some left over pieces. (Of course they can always count on as Lourdes did but here students are encouraged to figure it out without counting.) The purpose of this activity is to provide tasks where students encounter the need to mentally create another roll by packing up the pieces.
2. The individual pieces should be arranged in easily recognizable patterns. The reason is that unless the pieces are in easily recognizable patterns, children will revert to counting them by ones, which defeats the purpose of this activity. Further, by using recognizable patterns, the teacher creates the opportunity for children to use what they know about combining small numbers, including number combinations that make 10. Think back to the solutions Danielle and Max gave. We also know that children that have not yet developed all of these number relationships continue to have opportunities to do so as they engage in this activity. Take a look at this image, for example.



   Imagine if you saw this using the flashing approach. After easily seeing that there are 2 rolls, most of us would need to count the pieces to figure out how many there are. There are no obvious numerical groupings to use to figure out what to combine to make another roll.
3. Since the crux of this activity relies on being able to mentally create another roll, the critical part of the task is that there are more than 10 pieces. The number of *rolls* in the initial configuration is not relevant. So, it doesn't make the task more conceptually demanding to have, say, 8 rolls rather than 3 rolls. In other words, just because a quantity is larger doesn't make it harder. Compare the two images shown here.





Both images shown here show the same number of pieces and in the same patterns. So the conceptual demands of creating another roll, which is the primary purpose of these tasks, are the same. Simply including more rolls, as in the first task, does not contribute to the purpose for which this task is designed. In other words, just making the quantity larger does not increase the learning opportunities the task provides.

4. Children can become confused if we use the word "candies" to refer to both the total quantity and to the individual loose pieces of candy. For this reason, we suggest referring to the loose pieces of candy as "pieces" rather than as "candies." So, for example, when the teacher asks, "How many candies does Mrs. Wright have?" the children understand that she means the total quantity not the individual pieces.
5. The last consideration is this. Rather than create drawings to use for candy flash, teachers may wish to use Unifix cubes instead. Here you see the same example we showed earlier but using Unifix cubes instead of a drawing. There are several advantages to using Unifix cubes and at least one disadvantage.

Advantages of using Unifix cubes

- One advantage is that it is very easy to create the tasks right at the moment. The teacher does not need to create drawings in advance. (Of course, she can always draw them at the moment using a pen or marker.)
- A second advantage is that children will easily relate these to the scenario because of their work with Unifix cubes in prior tasks.

- The third advantage is that as children explain how they combined pieces to create another roll, the teacher can move the pieces around to show their thinking. For example, the teacher can demonstrate Danielle's solution by actually moving four pieces from the left over to the right so they will be beside the 6 pieces already there.
- A fourth advantage is that the children can *see* the individual pieces within a roll. For those children that might still be working hard to think of a roll as a single thing that is made up of ten pieces (ten other things), this image might be useful.

Disadvantage of using Unifix cubes

- This ability to *see* the individual pieces that comprise the roll is also a disadvantage in the following sense. With the individual pieces visible, a child can attempt to figure out the total quantity by counting everything by ones, including the ones that comprise the rolls. In our view this disadvantage does not outweigh the advantages. Nevertheless, we chose to show all of the candy flash pictures we have used so far using drawings precisely because the 10 individual pieces that make up the roll are not visible.

In the earliest stages of using this candy flash activity children often have very specific ideas of exactly *which* pieces they moved in their mind. For example, if the teacher were to ask Danielle *which* of the four pieces she moved, she might say, "I moved all of them except the one in the middle." Another child that also thought of taking 4 of the 5 to combine with the six might say that he moved a different set of 4. Eventually children will not be specific or if the teacher asks, "Which ones did you move?" the child might answer, "It doesn't matter." We interpret a comment such as this as indicating an advance in the child's reasoning. Now the child is reasoning quantitatively—take 4 from 5 to combine with 6 to get 10. The child is no longer thinking about packing up candies to make a roll.

Transformation Activity

Now we turn to the transformation activity we mentioned before. It is the last instructional activity in Part I of the CU instructional sequence. In this activity children are asked to think of different ways they can have a given quantity of candies.

> Example: Mrs. Wright has 43 lemon candies. Some of them might be packed up in rolls and some of them not yet packed up. Maybe none of them are packed up. Maybe they are all packed up. How many rolls and how many pieces might there be? Draw pictures to show different ways she can have 43 candies. Come up with as many different ways as you can.

Children work individually on this task and have paper and pencil to record the ways they think of. After children have had enough time to complete the task, there is a class discussion.

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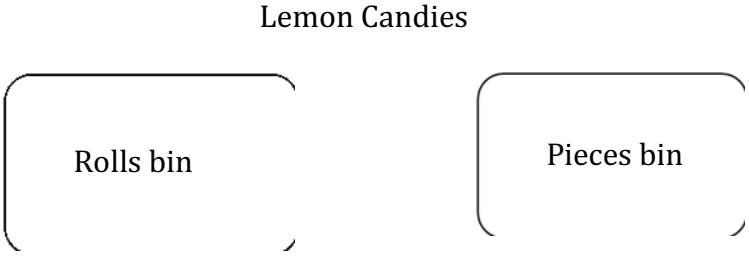
The purpose of this task is to create a situation in which students conceptualize a given quantity in terms of rolls and pieces in many different ways. In particular, while it is typical for students to think of a quantity of 43 candies as 4 rolls and 3 pieces (sometimes called the canonical form) or as 43 pieces, the goal of this activity is to elicit other ways as well, such as 2 rolls and 23 pieces, 3 rolls and 13 pieces, and 1 roll and 33 pieces. For children to have a deep conceptual understanding of place value it is imperative that they extend their thinking beyond these two "typical" ways to include these others as well.

Let me make a pragmatic point here. Some children will complete the task quickly while others will take much longer. Those children that complete the task quickly can be given another task, another number of candies. They can turn their paper over and complete this task on the other side. Each child should always have some meaningful mathematical activity to engage in at all times during the lesson. They should never be idle just waiting around for others to complete a task. Of course, the class will discuss only the task everyone worked on.

Introducing the transformation activity.
Many teachers have told us that they find it difficult to explain this task adequately. One way to introduce the task is to explain that for each flavor of candy, Mrs. Wright keeps a bin (tub) for rolls and another bin (tub) for individual pieces. So for cherry candies she has one bin for rolls and one bin for pieces. For lemon candies she has one bin for rolls and one bin for pieces. She has this for every flavor of candies.



When the workers first make candies they put all of the candies in the pieces bin. When they have time, they work at packing up the candies. When they have a roll packed up they put it into the rolls bin. So when Mrs. Wright comes to see how many candies of a certain flavor she has, there might be some in the rolls bin (these are the ones that have been packed up) and some in the pieces bin (these are the ones that have not yet been packed up).

Now pose the specific task for children to work on:

> Mrs. Wright's workers made 43 lemon candies. Draw pictures to show different ways to have 43 candies. Come up with as many ways as you can.

What do children do?
The drawings themselves are quite revealing. Here you see two different children's drawings for the task "draw different ways you can have 60 candies."

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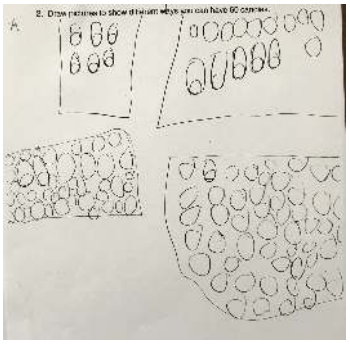
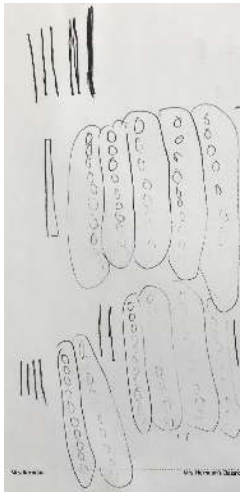


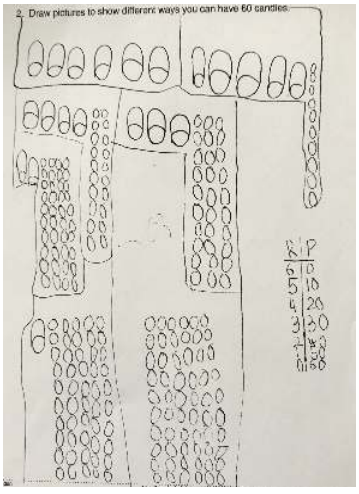
Lindsey's paper Amber's paper

If we look carefully we can see that Lindsey has drawn 5 rolls and 10 pieces of candy below the rolls. Significantly, to draw the rolls she did not draw ONE thing but clearly drew 10 different pieces that are connected to each other. Lindsey's drawing shows that she was trying to make her picture look like Unifix cubes. Children that make drawings like Lindsay did, typically think of a roll in terms of the *ones* (the pieces) that comprise it rather than as a single entity in and of itself.

Amber's picture, on the other hand, shows a roll as a single entity in her drawing. The first way Amber drew showed 6 rolls. The drawing below that one shows 2 rolls surrounded by 40 pieces. Without having the opportunity to watch her create the drawing or to ask her, we do not know exactly how she proceeded. But I have seen children create such a drawing by drawing the two rolls and then counting on from 20 by ones 21, 22, 23, all the way up to 60 as they draw the individual pieces. Such children typically do not know how many pieces they have drawn. If we were to ask them, they might need to go back to count the pieces by ones 1, 2, 3, 4, … up to 40, to figure out that they have 40 pieces.

Contrast these drawings to those of Lucas and Griffin shown below.





Lucas' paper Griffin's paper

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Lucas used a single line to indicate a roll. We can infer that Lucas thinks of a roll as a single entity—one thing. In fact, rather than drawing a picture of a roll, Lucas drew something to *represent* a roll. He drew circles to show the individual pieces but, interestingly, he has them arranged in groups of 10 and even has a loop around each group of 10. Without having the chance to ask him we do not know how he was thinking. Was he thinking that each group of 10 pieces could be packed into a roll or maybe that he unpacked a roll to get those 10 pieces? In any event, his drawing reveals a different type of organization from Amber's. For example, if we were to ask Lucas how many pieces he has in the second way he drew, he would most likely easily say that he has 50 pieces. He might say, "10, 20, 30, 40, 50" or simply "50." But he almost surely would not count the piece by ones from one to answer our question.

Griffin's paper shows all of his work for this task. We can see that he was systematic in his drawings. First he shows 6 rolls, next 5 rolls and 10 pieces, then 4 rolls and 20 pieces, 3 rolls and 30 pieces, 2 rolls and 40 pieces, 1 roll and 50 pieces, and 60 pieces. These drawings are clearly marked off from each other and in each case he drew the pieces in columns of 10. While his drawings were systematic, indicating quite sophisticated reasoning, the way Griffin drew the rolls indicates that he was still thinking very much in terms of the rolls in Mrs. Wright's candy shop, unlike Lucas who drew a single line to represent a roll. In addition, Griffin summarized his results in a table.

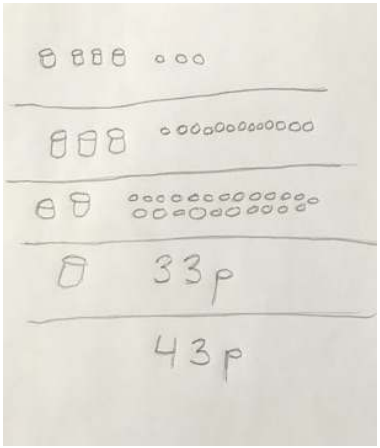
Shifts in students' reasoning. These children's drawings show the variety in sophistication that presumably reflects the way each child is thinking about the task. We know that as students work on tasks of this type several shifts in their reasoning might occur. One shift is from creating different arrangements of the given quantity as unrelated configurations of rolls and pieces to viewing each arrangement as the result of a transformation from a previous one.

We can see another shift in the type of justifications students give as they explain their results to the class. For example, initially students might justify their way of showing the quantity either by counting their collection (to verify that it is the given quantity) or by describing how it can be packed up to be in the canonical form. In this case the justification is based on *empirical* reasoning. Here is an example. If a student answered 3 rolls and 13 pieces as a way to show 43, he might count the rolls as 30 and then the pieces (31, 32, 33, 34, … 43) for a total of 43. Or he might reason that 10 pieces of the 13 can be packed into a roll yielding a total of 4 rolls and 3 pieces (the canonical form).

A shift in reasoning is indicated when students reason that a configuration *has to be* correct because it is equivalent to a previously agreed on configuration. In this case the justification is based on *logical* reasoning. For example, if the class has already agreed that 3 rolls and 13 pieces is one way to have 43, a child might justify his way of 2 rolls and 23 pieces by saying that if we have 3 rolls and 13 pieces we can unpack one of the rolls to yield 2 rolls and 23 pieces. Notice that in this example the child reasoned that his way is correct because it is quantitatively *equivalent* to another way that we already agreed was correct.

Another shift we have observed frequently is from drawing pictures of rolls and pieces to using numerals to indicate the number of rolls and pieces. In particular, students often make this shift when the number of pieces in their configuration is large. For example, if a student wants to show 1 roll and 33 pieces, it is time consuming to draw all of the 33 individual pieces. Further, when the student later wants to report what he has during a subsequent class discussion, he frequently has to go back and recount the pieces to find out that he has 33. Of course, some students will "just know" that they have 33 pieces because of logical reasoning. Again, these differences in students' ways of reasoning and explaining reveal advances that some of them have made. A goal is that students are able to reason about the quantities without having to count them one by one.

Xavier's paper, seen below, shows an example of making a shift to using numerals.



Xavier's paper for "ways to show 43 candies"

Xavier's paper shows that he started by using drawings but when the number of pieces became large he shifted to using numerals. He was systematic in the way he organized the ways. Our experience is that for children to shift to using numerals in the way that Xavier did, they have to already know how many pieces there will be in the way they want to show. For example, here Xavier knew that he needed 33 pieces to accompany one roll.

The transformation task is appropriately used for several lessons with one or two tasks per lesson, with a thorough discussion of each task. We remind teachers that our goal is never that students learn *to do* an activity. Our goal is not that they become proficient at the activity. Rather, these activities are designed to promote conceptual development. So we caution teachers to not use an activity over and over until students *get good* at it. Rather, the teacher should pay attention to how children reason and to the shifts they make in their reasoning, the shifts in reasoning in the class as a whole and the shifts individual children are making.

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At the same time, we can see from the differences in Lindsey's work and that of Lucas and Griffin that there are big differences in their conceptual understandings of tens and ones. With this in mind, the teacher might decide to use the transformation task once or twice later on in the sequence, perhaps while already using activities from Part II, as a way to give students like Lindsey an opportunity to reason about transformations after they have had more experiences with using the Unifix cubes to solve problems posed in the candy shop setting.

A typical lesson at this time in the sequence might begin with two or three *Candy Flash* tasks followed by several *transformation tasks*. The limiting factor about how much to include in a lesson is how much time is needed for thorough discussions of the tasks used.